Tensor Products With No Holes

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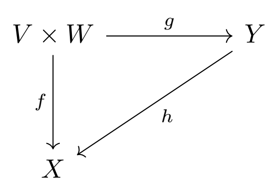
Let k be a field, and let V, W be vector spaces over k (and unless otherwise specified, this will be the ground field for all other vector spaces discussed onward). A tensor product of V and W is a pair (X, f), where X is a vector space and f is a bilinear map $V \times W \to X$, satisfying the following "universal" property: for any bilinear map $g : V \times W \mapsto Y$ into another (possibly the same) vector space Y, there exists a unique linear map $h : X \mapsto Y$ such that $f = h \circ g$. This is to say that every bilinear map from $V \times W$ into any arbitrary vector space "factors through" X.



We now point a couple things out. Firstly, the definition alone says nothing about whether or not such an object exists. In a moment, however, we will provide a working construction. Secondly, it turns out that if (X_1, f_1) and (X_2, f_2) are two tensor products of V and W, they are guaranteed to be the same up to "unique isomorphism". What that means is that we can guarantee there exists an isomorphism $F : X_1 \to X_2$ such that $f_2 = F \circ f_1$ and $f_1 = f_2 \circ F^{-1}$, and moreover, we can guarantee that this isomorphism is unique. Because of this, it is common to speak of *the* tensor product between V and W, as any two different tensor products will have a unique isomorphism between them. Well actually, I must admit that the last sentence was not at all obvious to me when first learning about the tensor product. Sure, if two groups are isomoprhic, then one is just a "relabeling" of the other and thus they are for all intensive purposes the same. But in linear algebra, all vector spaces of the same dimension are isomorphic to each other (and thus we care about more), so existence of an isomorphism alone doesn't really tell us much. Thus it must be the "unique" part in "unique isomorphism" that is really important, and indeed it is. I can't tell you any more though, I'm just as confused.

The standard convention is to denote the tensor product of V and W as $V \otimes W$, and so we do onwards. Moreover, it is also convention to denote the corresponding bilinear map as $\otimes$, and the function evaluation $\otimes(v, w)$ as $v \otimes w$. That last sentence can lead to initial confusion, but makes the notation a lot easier going forward (but it takes a long time to get an inuitive feel for why this choice is made, so don't beat yourself up too much if you are confused at first), so take a moment to internalize it. Let's now prove the previously stated "unique up to unique isomoprhism property". If (X_1, f_1) and (X_2, f_2) are two tensor products of V and W, then the universal property gives us unique maps $g_1 : X_2 \to X_1$ and a map $g_2 : X_1 \to X_2$ so that $f_1 = g_1 \circ f_2$ and $f_2 = g_2 \circ f_1$. One deduces that $g_1 = g_2^{-1}$, and moreover uniqueness is already established, so we are done.

Now for the construction. We will provide two constructions, and this first one will be require a choice of basis and assumes that V and W are finite dimensional, while the second one does neither. The first is included to provide more intuition as to what a tensor product actually looks like.

Let $\mathcal{B}_V = \{v_1, \ldots, v_m\}$ and $\mathcal{B}_W = \{w_1, \ldots, w_n\}$ be the two respective bases for V and W. Then, the set of formal linear combinations[1] (where scalars are taken from the field k) of the mn elements $\{v_i \star w_j \mid v_i \in \mathcal{B}_V, w_j \in \mathcal{B}_W\}$ is the tensor product $V \otimes W$, with the bilinear map given by $\otimes(v, w) = v \star w$. Effectively, $v \star w = v \otimes w$. If $g : V \times W \to Z$ is a bilinear map into a vector space Z, we must show there exists a unique $h : Z \to V \otimes W$ such that $\otimes = h \circ g$. All we have to do is make h send $g(v_i, w_j) \mapsto v_i \otimes w_j$, and the rest of the map becomes determined by linearity. Thus this construction is indeed the tensor product.

For our second construction, start with the free vector space U on $V \times W$. That is, elements of U are (unique linear combinations) of the form

$$\sum_{(v,w) \in V \times W} c_{(x,y)}(x, y)$$

where $c_{(x,y)} \in k$ are scalars, and all but finite of them being nonzero (linear combinations must be a sum of a finite number of vectors). Now let $U_0 \subset U$ be the subspace spanned by elements of the form

$$(x, y) + (x, y') - (x, y + y')$$

$$(x, y) + (x', y) - (x + x', y)$$

$$\lambda(x, y) - (\lambda x, y)$$

$$\lambda(x, y) - (x, \lambda y)$$

for $x, x' \in V$, $y, y' \in W$, and $\lambda \in k$. Quotient U out by this set gives us the tensor product. That is, $U/U_0 \cong V \otimes W$.

We start out with the large vector space $U = \{v \otimes w \mid v \in V, w \in W\}$ (before we used $\star$ for clarity, but now we are beginning with $\otimes$ and thus implicitely defining the bilinear map f). Let $U_0 \subset U$ be the subspace spanned by the

[1] this is known as the free vector space on the set $\mathcal{B}_V \times \mathcal{B}_W$

elements

$$\begin{aligned}
U_0 = &\langle \lambda v \otimes w - \lambda(v \otimes w), \\
&v \otimes \lambda w - \lambda(v \otimes w), \\
&(u+v) \otimes w - u \otimes w - v \otimes w \\
&u \otimes (w+x) - u \otimes w - u \otimes x \rangle
\end{aligned}$$

taken over all $u, v \in V$, $w, x \in W$, and $\lambda \in k$. If we quotient U under this set (or "mod out the relations"), the resulting set is the tensor product, or $V \otimes W = U/U_0$.