Munkres Notes I Guess

Let X be a set, and let $\tau \subset \mathcal{P}(X)$ be a collection of subsets of X. We say that τ is a **topology** on X if the following three conditions are satisfied:

1. $\emptyset$ and X are in τ.
2. The union of the elements of any subcollection of τ i in τ.
3. The intersection of the elements of any finite subcollection of τ is in τ.

The tuple (X, τ) is known as a **topological space**, but when contex is clear, we manytimes refer to X as a topological space. The members of τ are called **open sets**. The topology of all subsets of X is known as the **discrete topology** and the topology $\tau = \{\emptyset, X\}$ is known as the **indiscrete topology**.

The **finite complement topology** τ_f is the collection of all subsets U of X such that $X - U$ is either finite or all of X. Why is this a topology? Clearly $\emptyset \in \tau$ since $X - \emptyset = X$, and $X \in \tau$ since $X - X = \emptyset$ is finite. Moreover, if $\{U_\alpha\}$ is an indexed family of nonempty elements of τ_f, then by De Morgan's Law,

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

is finite since each $X - U_\alpha$ is finite, so $\bigcup U_\alpha \in \tau_f$. Moreover, if $U_1, \ldots, U_n$ are nonempty elements of τ_f, then

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

which is a finite union of finite sets, hence finite, so $\bigcap_{i=1}^{n} U_i \in \tau_f$. In the case that we chose empty elements for any of the U_i or U_α, the RHS would either be all of X or what it would be with the empty elements removed from the family.

Suppose that τ and τ' are two topologies on a given set X. Then we say that τ' is **finer** than τ if $\tau' \supseteq \tau$; if τ' *properly* contains τ, then we say that τ'

is **strictly finer** than τ. We also say that τ is **coarser** and **strictly coarser** than τ' in these two respective situations. We say that τ is **comparable** with τ' if either $\tau' \subset \tau$ or $\tau \subset \tau'$. Note that the collection of all topologies on a set forms a partially ordered set by inclusion, and *not* a totally ordered set totally ordered set since not all topologies are comparable. Sometimes we use the terms **larger** and **smaller** instead of finer and coarser, respectively.

It is convenient to explicitely specify all open sets of a topology τ. Fortunately, we have a way of specifying a small number of subsets of X and definining a topology in terms of them.

A **basis** for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X (called **basis elements**) such that:

1. For each $x \in X$ there is at least one basis element B containing x.

2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, we define the **topology τ generated by $\mathcal{B}$** as follows: a subset U of X is said to be open in X (an element of τ) if for each $x \in U$ there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$. Note that each basis element is itself open in τ.

Let's now prove that the collection of subsets specified above is actually a topology. $\emptyset$ is vacously open, and X is open since for every $x \in X$ there exists a $B \in \mathcal{B}$ such that $x \in B \subset X$. If $\{U_\alpha\}$ is a collection of open sets, then for the set

$$U = \bigcup U_\alpha,$$

for every $x \in U$ there exists a U_α containing x, and since U_α is open, there exists a $B \in \mathcal{B}$ such that $x \in B \subset U_\alpha \subset U$, so U is open. Moreover, if U_1 and U_2 are two open sets, then for the set $U_1 \cap U_2$, for every $x \in U_1 \cap U_2$, there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By property (2) we are guaranteed the existence of a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$, and since, $B_1 \cap B_2 \subset U_1 \cap U_2$, we get that $U_1 \cap U_2$ is an open set. Now, for open sets $U_1, \ldots, U_n$, we can easily inductively extend this to show that $U_1 \cap \ldots U_n$ is open.

It is important to note that if τ is the topology generated by a basis $\mathcal{B}$, then τ equals the collection of all unions of elements of $\mathcal{B}$. Why? If U is an open set in the topology generated by $\mathcal{B}$, then for each $x \in U$ there exists a $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$, so $U = \bigcup_{x \in U} B_x$ is a union of elements of $\mathcal{B}$. Conversely, if $U = \bigcup B_\alpha$, clearly U is open (in the topology generated by $\mathcal{B}$) since it is the union of a collection of open sets.

We have described how to go from a basis to the topology it generates. How about the reverse direction: from a topology to a basis generating it? We answer this question below.

For a topological space X, suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X. Why? We first must show that $\mathcal{C}$ is actually a basis, and then that the topology τ' generated by $\mathcal{C}$ is equal to τ. The first condition of being a basis is satisfied since for the open set X, for each $x \in X$ we are told there exists a $C \in \mathcal{C}$ such that $x \in C \subset X$. Moreover, if $C_1, C_2 \in \mathcal{C}$, the set $C_1 \cap C_2$ is open since C_1 and C_2 are open, so by hypothesis, for every $x \in C_1 \cap C_2$ there exists a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$. Thus this is a valid basis. Does it generate τ though? That is, is the topology τ' generated by $\mathcal{C}$ equal to τ? Let U be an open set of X. Then for each $u \in U$ there exists $C_u \in \mathcal{C}$ such that $u \in C_u \in U$, so $U = \bigcup_{u \in U} C_u$ and is thus open in τ' as well. Conversely, suppose that U is open in τ'. Then $U = \bigcup C_\alpha$ for some family of basis elements $\{C_\alpha\}$ (we just proved this). This is a union of open sets in X, and is thus open in τ.

Given two topologies on a set and their respective bases, how should we go about comparing if one is finer than the other?

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ', respectively, on the set X. Then the following are equivalent:

1. τ' is finer than τ.
2. For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there exists a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

($\implies$): If we are given an $x \in X$ and a $B \in \mathcal{B}$ containing x, then B is also open in τ', so there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.
($\impliedby$): This means that every $B \in \mathcal{B}$ is open in τ', so clearly τ is contained in τ'.

Note this means that the topology on $\mathbb{R}^2$ generated by the basis of open balls is the same as the topology generated by rectangles.

The **standard topology** on the real line is the topology generated by all

$$(a, b) = \{x \mid a < x < b\}.$$

Let X, Y be topological spaces and let $f, f' : X \to Y$ be continuous maps. We say that f is **homotopic** to f' if there is a continuous map $X \times I \to Y$ such that

$$F(x,0) = f(x) \qquad \text{and} \qquad F(x,1) = f'(x)$$

for each x. (Here $I = [0,1]$). The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nullhomotopic**.

Recall that a **path** from point x_0 to point x_1 is a continuous map $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We call x_0 the **initial point** of the path and x_1 the **final point**.

Two paths f and f' are said to be **path homotopic** if they have the same inital and final point x_0 and x_1, and if there exists a continuous map $F : I \times I \to X$ such that

$$F(s,0) = f(s) \qquad \text{and} \qquad F(s,1) = f'(s)$$

$$F(0,t) = x_0 \qquad \text{and} \qquad F(1,t) = x_1$$

We call F a **path homotopy** between f and f' and write $f \simeq_p f'$. The first condition says that F is a homotopy between f and f', and the second condition says that each path $f_t(x) = F(x,t)$ is from x_0 to x_1. In other words, the first condition says that F represents a continuous way of deforming the path f to the path f', and the second condition says that the end points of the path remain fixed during the deformation.

We now show that $\simeq$ and $\simeq_p$ are equivalence relations:
For a given f, $F(x,t) = f(x)$ is a homotopy from f to f. It is continuous as it is the projection map of f from $X \times I$ into X. If f is a path , F also satisfies the second path-homotopy condition and is thus a path homotopy as well.
Given $f \simeq f'$, we show that $f' \simeq f$. If F is the between f and f', then $F'(x,t) = F(x, 1-t)$ is a homotopy between f' and f. If F is a path homotopy, then so is F'.
If $f \simeq f'$ and $f' \simeq f''$, we will show that $f \simeq f''$. Let F be a homotopy between f and f', and let G be a homotopy between f' and f''. Then the function

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{1}$$

is a homotopy between f and f''. It is well defined at $x = \frac{1}{2}$ as $F(x,1) = f'(x) = G(x,0)$, and continuity follows from the pasting lemma. If F and G are path homotopies, then it is easy to see that H is also a path homotopy.