Math 55b: Studies in Analysis and Topology

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Abstract

This is a math course where we cover a lot of math.

1 January 23rd (Lecture 1)

First half of the semester is topology, the second is devoted to complex analysis. We will aim to show the deep connection between these two subjects (and Harris even claims this connection is even greater than that between real and complex analysis).

Topology is the study of geometric objects invariant under continuous transformations.

The definition of a topological space is not well motivated (unlike how we started with groups in 55a), so we first will discuss metric spaces.

A **metric space** is a set with a map $\alpha : X \times X \to \mathbb{R}^{\geq 0}$ such that $d(p, q) = d(q, p),\ d(p, q) = 0 \Leftrightarrow p = q,\ d(p, r) \leq d(p, q) + d(q, r)$.

For example, in $X = \mathbb{R}^n$, the **standard metric** is $d(x, y) = \left(\sum (x_i \cdot y_i)^2\right)^{1/2}$, and there are others too. The **diamond metric (or Euclidean)** is $d(x, y) = \sum |x_i - y_i|$, and the **square metric** is $d(x, y) = \max\{x_i - y_i\}$. These get their names from the locus of points that are equidistant from a given point. It is a good exercise to check that these are valid metrics.

There is also the notion of the **induced metric**. If (X, α) is a metric space, and $Y \subset X$ is any subset, the induced metric $e : Y \times Y \to \mathbb{R}$ is just the restriction $e = \alpha|_{Y \times Y}$. Similarly, given two metric spaces $(X, d), (Y, e)$, the product metric on the set $X \times Y$ is given by

$$d((x, y), (x', y')) = \sqrt{d(x, x')^2 + e(y, y')^2}$$

Suppose $(X, d), (Y, e)$ are metric spaces, $f : X \to Y$ a map of sets, and $p \in X$. We say that **f is continuous at p** if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$d(p, q) < \epsilon \implies e(f(p), f(q))$$

Let $p_1, p_2, p_3, \dots \subset X$. We say that $p_n \longrightarrow p$ "**converges to p**" if $\forall \epsilon > 0, \exists N$ such that $\forall n \geq N$, $\alpha(p, q) < \epsilon$. Another way to say this is as follows. Set

$$Y = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\} \subset \mathbb{R}$$

we can define a map $Y \to X$ such that

$$\frac{1}{n} \mapsto p_n$$
$$0 \mapsto p$$

$p_n \longrightarrow p \Leftrightarrow Y \to X$ is continuous.

Observation: given $p_1, p_2, p_3, \cdots \subset X$, the limit, if it exists, is unique. If we had $p_n \longrightarrow p$ and $p_n \longrightarrow q$, then when $\epsilon = \frac{1}{2}\, d(p,q)$, we have $\forall n \geq N$,

$$d(p_n, p) < \frac{1}{2} d(p,q)$$

$$d(p_n, q) < \frac{1}{2} d(p,q)$$

Adding these violates the triangle inequality.

Let (X, d) be a metric space, and let $U \subset X$. We say that U is **open** if $\forall p \in U$, $\exists \epsilon > 0$ such that $\forall q \in X$ $d(p,q) < \epsilon \implies q \in U$. For example, in $\mathbb{R}$, (a,b) is open, and $[a,b], (a,b]$ are not open.

There is an easier way to define open sets.

$\forall p \in X$, and any $r > 0$, define the **ball with radius r around p** as

$$B_r(p) = \{q \in X : d(p,q) < r\}$$

As an exercise, use the triangle inequality to verify that all balls are open subsets. Now we can say that a set $U \subset X$ is open if $\forall p \in U$, $U \supset B_r(p)$ for some $r > 0$. Or equivalently,

$$U \text{ open} \;\Leftrightarrow\; \text{U is a union of open balls}$$

As a consequence, any arbitrary unions of open sets in X are also open. However, this is *not* true for infinite intersections (it is for finite intersections by De Morgan's Law), for example in $\mathbb{R}$, $\bigcap B_{1/n}(0) = \{0\}$.

Key theorem: If $f : X \to Y$ is a map of metric spaces, it is continuous if and only if all open sets in the target have preimage which are also open. That is, if the statement "$U \subset Y$ is open $\implies f^{-1}(U)$ is open in X" for all open subsets U, then this is equivalent to saying that f is continuous.

Proof: Let $p \in X$ be any point. Recall that f is continuous at p if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$d(p,q) < \delta \implies e(f(p), f(q)) < \epsilon$$

but this is equivalent to

$$f^{-1}\left(B_\epsilon\left(f(p)\right)\right) \supset B_\delta(p)$$

Similarly, we can say that the sequence of points $p_1, p_2, p_3, \cdots \in X$ converges to p if and only if every open set $U \subset X$, that contains some point in the sequence, contains all but finitely many p_n.

Now we are ready to define topological spaces. A **topological space** is a set X along with a subset $U \subset 2^X$ (the powerset $\mathcal{P}(X)$) such that arbitrary unions of open sets are open, same with finite intersections, and lastly, $U \supset \emptyset, X$.

We say that a map $X \to Y$ between topoligcal spaces is **continuous** if... until next time.

Check as an exercise that all metric spaces are topological spaces. That is, there is a mapping

$$\{\text{metric spaces}\} \longrightarrow \{\text{top. spaces}\}$$

but it is neither injective (the metric spaces in $\mathbb{R}^n$ we talked about all correspond to the same topological space) nor surjective (not all topological spaces are metrizable).

The "extreme" topologies on a set X are $U = \mathcal{P}(X)$ and $U = \{\emptyset, X\}$, the **discrete** and **indiscrete** topologies. Note that only the first is metrizable, with an example metric being

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases} \tag{1}$$

That is the end of the lecture!