Math 55a Notes

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1 9/16/22, Friday

If G is an abelian group, and $\alpha \in G$, there exists a homomorphism $\mathbb{Z} \to G$ $(n \mapsto n\alpha)$. There are two possibilities for α: if the map is injective, then α has infinite order, Else, $\ker(\phi) = n\mathbb{Z}$ where n is the smallest positive integer such that $n\alpha = 0$. We say α has order n.
IF k is a field, there exists a field (ring) homomorphism $\phi : \mathbb{Z} \to k$ $(n \mapsto n \cdot 1 = 1 + \cdots + 1)$. If ϕ is injective, we say k has characteristic 0 (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$). If $\ker(\phi) \neq \{0\}$, observe that the order n of an element must be a prime p. Field homomorphisms preserve characteristics.
Let k be a field, V a vector space with respect to k is defined by the operations $V \times X \to V$ and $k \times V \to V$.
For example, $k^n = \{(a_1, \dots, a_n) : a_i \in k\}$ is a vector space over k for any positive integer n.
More generally, if S is any set, then the set $\{\text{maps } S \to k\}$ is a vector space. In the example above, we can think of $S = \{1, \dots, n\}$. If $S = \mathbb{N}$, then $\{\text{maps } S \to k\} = k[[x]]$, the "set of power series in k".
We define $W \subset V$ as a subspace if it is closed under additional and scalar multiplication. Observe that given two subspace W, W', then $W \cap W'$ is also a subspace.
We define a vector space homomorphism (linear map) of two vector space over k as a map $\phi : V \to W$ that respects $+, \times$:

$$\phi(v + v') = \phi(v) + \phi(v')$$
$$\phi(\lambda v) = \lambda \phi(v)$$

Note that subfields form vector space over their "parent" field.
Let $S \subset V$, then we can define

$$\overline{S} = \text{ smallest subspace } W \subset V : W > S$$

, eg. if $S = \{v_1, \dots, v_n\}$, then $\overline{S} = \{a_1 v_1 + \cdots + a_n v_n \in V | a_i \in K\}$. This is called the **span** of S.
Definition: We say that $S \subset V$ is **linearly independent** if

$$a_1 v_1 + \cdots + a_n v_n = 0 \Leftrightarrow a_i = 0 \quad \forall i$$

We say that S is a **basis** for V if both are true: S is lineraly independent and $\overline{S} = V$.
Another way to say all of these things: given a set $S = \{v_1 \dots v_n\} \subset V$, then there exists a homomorphism

$$k^n \to \phi V$$
$$(a_1, \dots, a_n) \mapsto \sum a_i v_i$$

Now we can say that S **spans** V if ϕ is surjective, linearly independent if ϕ is injective, and a basis of ϕ is an isomorphism.
For example, if $V = k^n = \{(a_1, \dots, a_n) | a_i \in k\}$ has "standard basis"

$$e_i = (0, \dots, 0, 1, 0, \dots, 0),$$

where there is a 1 in the i^{th} position and 0 everywhere else. For example, if $k[x]$ has basis $\{1, x, x^2, \dots\}$, can you find a basis for $k[[x]]$ (hint: you need the axiom of choice) ?
Observe, if $S = \{v_1, \dots, v_n\} \subset V$ is a finite spanning set, then $\exists S' \subset S : S'$ is a basis.
The condition $v_1, \dots, v_n$ is not a basis means

$$\exists (a_1, \dots, a_n) \neq 0 : a_1 v_1 + \dots a_n v_n = 0$$

Say $a_j \neq 0$, then

$$v_j = \frac{-1}{a_j} \left(a_1 v_1 + \dots + a_n v_n \right).$$

This shows that v_j is a linear combination of $\{v_i\}_{i \neq j}$. We can keep removing these linear dependence relations until we get a basis.
Proposition: If V is a vector space, and two bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$, then $m = n$, so we can say V has **dimension** n.

Proof. If S is a basis for V,

$$S' \subsetneq S \Rightarrow S' \text{ is lin. ind. but doesn't span}$$

$$S \subsetneq S' \Rightarrow \text{ spans, but not lin. ind.}$$

This is the same as saying the proper subset or proper superset of a basis is not a basis. We now claim, ... I give up this is the same proof from Axler. □

Given vector spaces V, W over k, then we define

$$V \times W := \{(v, w) : v \in V, W \in W\}$$

where the operations are defined in their obvious way (component-wise). Observe that $\dim V \times W = \dim V + \dim W$, as $(v_1, 0), \dots, (v_m, 0), (0, w_1), \dots, (0, w_n)$ is a basis for $V \times W$.
Define

$$\text{Hom}(V, W) = \{\phi : V \to W : \phi \text{ is a homomorphism}\}$$

where

$$\phi, \psi : V \to W$$

$$(\phi + \psi)(v) \mapsto \phi(v) + \psi(v)$$

$$(\lambda \phi)(v) = \lambda \phi(v)$$

Notice that

$$\dim \text{Hom}(V, W) = \dim V \cdot \dim W$$

We now define V^*, the **dual** vector space of V, as

$$V^* = \{\text{linear maps (hom.)} : V \to k\} = \text{Hom}(V, k)$$

Claim: Let $v_1, \ldots, v_m$ be a basis for V, and $w_1, \ldots, w_n$ for W. Then $\forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ define

$$\phi_{ij}(v_i) = w_j$$

$$\phi_{ij}(v_k) = 0 \quad k \neq i$$

Show that ϕ_{ij} forms a basis for $\text{Hom}(V, W)$

2 9/19/22, Monday

Review: Any linearly independent subset $S \subset V$ can be enlarged to a basis. Any spanning set contains a basis.
Proposition: For a linear map $\phi : V \to W$ from a finite dimensional vector space to another finite dimensional vector space, we have that

$$\dim \ker(\phi) + \dim \text{im}(\phi) = \dim V$$

Proof. Say $\dim V = n, \dim W = n$, choose a basis $v_1, \ldots, v_k$ for $\ker(\phi) \subset V$, and enlarge this basis to $v_1, \ldots, v_m \in V$. The crucial claim is that $\phi(v_{k+1}), \ldots, \phi(v_m)$ form a basis for $\text{im}(\phi)$. Prove this as an exercise, or see Axler. □

Define **rank**$(\phi) := \dim \text{im}(\phi) = m - \dim \ker(\phi)$, then the above can be restated as the rank plus the "nullity" is the dimension of the domain for a linear map.

Given $V \subset W$ over a field K, we can form the **quotient space**

$$W/V := \text{ quotient as abelian group}$$

with the operation

$$k \times W/V \to W/V$$

for scalar multiplication. For an example, think of $V = \mathbb{R}^2$ and $W \subset V$ as the line $y = x$. Then the set of cosets of W in V are all the lines parallel to W. Note that (prove that): $\lambda \cdot \overline{V} = \overline{\lambda V}$.
Note that we have a similar correspondence as we did with groups:

$$\{\text{subspaces of } W/V\} \Longleftrightarrow \{\text{subspaces of } W \text{ that contain } V\}$$

Now we talk about direct sums/products. Say $V_1, \ldots, V_n$ are vector spaces over K, then we can define

$$\bigoplus V_\alpha = \prod V_\alpha = \{(v_1, \ldots, v_n) : v_\alpha \in V_\alpha\}$$

Side Note: If there is an ∞ collection of V_α, then $\prod V_\alpha = \{(v_1, v_2, \ldots) : v_\alpha \in V_\alpha\}$ and $\bigoplus V_\alpha = \{\text{same thing but all but finitely many } v_\alpha = 0\}$.

If $V_1, \dots, V_n \subset W$ are subspaces, then we say $V_1, \dots, V_n$ **span** W if any $w \in W$ can be written as

$$w = a_1 v_1 \cdots + a_n v_n, \quad v_\alpha \in V_\alpha$$

We add the additional condition that these vectors must be linearly independent, i.e. $v_1 + \cdots + v_n = 0 \Rightarrow v_i = 0 \ \forall i$, or equivalently,

$$a_1 v_1 + \cdots + a_n v_n = 0 \, (v_i \in V_i, v_i \neq 0) \Rightarrow a_1 = \dots a_n = 0$$

If $V_1, \dots, V_n \subset W$ are subspaces that span and are linearly independent, then we write $W = \bigoplus V_\alpha$, i.e. every $w \in W$ can be uniquely expressed as a sum of values $v_\alpha \in V_\alpha$. In this case, $\dim W = \sum_\alpha \dim V_\alpha$.

Say V, W are finite dimensional vector spaces over K, where $\dim V = m, \dim W = n$. Recall that

$$\mathrm{Hom}(V, W) := \{\text{linear maps } \phi : V \to W\}$$

To describe $\mathrm{Hom}(V, W)$, we start with bases: $v_1, \dots, v_m$ basis for V, $w_1, \dots, w_n$ basis for W. Note that any linear map is determined by $\phi(v_i)$. Conversely, we can choose arbitrarily $\phi(v_1), \dots, \phi(v_m) \in W$ and get a (unique) linear map. Write

$$\phi(v_1) = a_{11} w_1 + \dots a_{n1} w_n$$

$$\vdots$$

$$\phi(v_m) = a_{1m} w_1 + \dots a_{nm} w_n$$

This means that $\mathrm{Hom}(V, W) \cong \{m \times n \text{ matrices}\} \cong K^{mn}$, where the matrix above is constructed as $\{a_{ij}\}$. Another way to say this: a basis $v_1, \dots, v_n$ for V is the same thing as an isomorphism $V \to K^n$. Insert commutative diagram now where canonical isomorphisms factor.

What happens if we choose a different basis $v'_1, \dots, v'_m$ for V¿ We get a different matrix representation A', i.e. write:

$$v'_1 = p_{11} v_1 + \cdots + p_{m1} v_m$$

$$\vdots$$

$$v'_m = p_{1m} v_1 + \cdots + p_{nm} v_m$$

An inverse of this map can be defined as well, by decomposing the v_α in terms of the v'_α. Set the original $m \times m$ matrix to be $A' = A \cdot P$. Likewise, if we choose a different basis $w'_1, \dots, w'_n$ for W, and we can write

$$w'_i = q_{1i} w_i + \cdots + q_{ni} w_n$$

to get an $n \times n$ matrix Q that is invertible. In this case, the new map is $A' = Q^{-1} A$. In general, by choosing a different basis for both sides V, W, we can replace the given matrix A with $Q^{-1} A P$ where Q is an invertible $n \times n$

matrix, P is an invertible $m \times m$ matrix.

Given a linear map $\phi : V \to W$, where V has dimension m and W has dimension n, start by choosing a basis $v_{r+1}, \ldots, v_m$ for $\ker(\phi)$. Here, $\text{rank}(\phi) = r$. We can enlarge this basis to $v_1, \ldots, v_m$ for V. Next, set $w_1 = \phi(v_1), \ldots, w_r = \phi(v_r)$, and we can complete this to a basis $w_1, \ldots, w_n$ for W. More concretely, we defined ϕ as

$$\phi : v_i \mapsto w_j \quad i = 1, \ldots, r$$

$$v_j \mapsto 0 \quad j > r$$

and it has matrix representation of $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix.

This implies that there are finitely many linear maps (up to isomorphism) between two finitely dimensional vector spaces. A more detailed explanation, can be seen through commutative diagrams where the canonical maps $V \cong K^m$ and $W \cong K^n$ factor with the matrix map from $K^m \to K^n$, but drawing diagrams is hard.
You may be wondering what the point to linear algebra is since we just classified all linear maps, but observe: if V is a vector space, a linear map $\phi : V \to V$ is called an **operator**. For operators/automorphisms, then, the classification of linear maps is not as immediate as what we just saw.

3 9/21/22, Wednesday

Fix a field K, and let V be a vector space over K. Then define the **dual space**

$$V^* := \{\text{linear maps } V \to K\}$$
$$= \text{Hom}(V, K)$$

To write a basis for the dual space, we define the linear forms e_i^* by $e_i \mapsto 1$ and $e_j \mapsto 0$ for $j \neq i$, i.e., $(x_1, \ldots, x_n) \mapsto x_1$. This is a basis for $(K^n)^*$, so the isomorphism of the dual space with V is not natural (depends on choice of basis / goes through K^n), but we still have $V \cong V*$. By contrast, the isomorphism $V \cong (V*)*$ is natural, and is defined by: $v \mapsto l(v)$. As an exercise, check this is injective.

Let $W \subset V$ be a subspace. Define the **annihilator** as

$$\mathbf{Ann}(W) := \{l \in V^* : l(W) = 0\} \subset V^*$$

Observe that linear forms on $V/W \leftrightarrow$ linear forms $l : V \to K : l(W) = 0$. Thus, $Ann(W) \cong (V/W)^*$. Also observe that if $\dim V = n$ and $\dim W = k$, then $\dim \text{Ann}(W) = n - k$.

Say $\phi : V \to W$ is a linear map, and $l \in W^*$ (i.e. $l : W \to K$), then $l \circ \phi : V \mapsto K$. This gives a map $t\phi : W^* \to V^*$, the **transpose** of ϕ. As an exercise, prove that $t(t\phi)) = \phi$.

We now relate this transpose to the idea of transpose from matrices. If you choose a basis $v_1, \ldots, v_m$ for V and $w_1, \ldots, w_n$ for W, then $V \cong K^m$ and $W \cong K^n$, and so it follows that $V^* \cong K^m$ and $W^* \cong K^n$.

Suppose $\phi : V \to W$ is a linear map and $l : W \to K$ is a linear form. When does the transpose, $l \circ \phi$, equal the zero function. This happens when $\ker(t\phi) = \{l : W \to K : l \circ \phi = 0\} = \text{Ann}(\text{im}(\phi))$.

We will now talk about polynomials over a field K. A polynomial is $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ (finitely many terms, degree n or less, forms a vector space). This defines a function $K \to K$ (obviously not linear).
First basic fact: if $f(\lambda) = 0$ for some $\lambda \in K$, then, f is divisible by $(x - \lambda)$, in other words, we can write $f(x) = (x - \lambda)g(x)$ for some $g \in K[x]$.

4 9/26/22, Monday

Wednesday: finish up operators, introduce bilinear forms
After that: multilinear algebra, more on groups, representation theory
Midterm exam: will be take-me, posted Wednesday 10/12, due Friday 10/14
Today: Brief intro to language of categories and functors, and more on operators

A **category** C consists of 3 things:

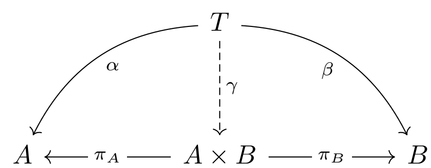
- a collection of **objects** $Ob(C)$
- for any $A, B \in Ob(C)$, a set of morphisms $Mor(A, B)$
- a law of composition: $\forall A, B, C \in Ob(C)$, a map
$$Mor(A, B) \times Mor(B, C) \to Mor(A, C)$$
- Associativity of composition: $\forall A, B, C, D \in Ob(C)$ where
$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$$
$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$
- $\forall A \in Ob(C), \exists id_A \in Mor(A, A)$ such that $\forall A \xrightarrow{\phi} B, \phi \circ id_A = \phi$ and $id_A \circ \phi = \phi$

Examples The category (sets) is just the category of sets with morphisms being maps between them. A variant of these are known as pointed sets where for all sets A and elements $x \in A$, there are objects = pairs(A, X) with morphisms $(A, x) \to (B, y)$ are the set of maps $\phi : S \to B$ where $\phi(x) = y$.

Another example are the nested pairs which maps specific subsets to each other $A, B \subset A$. Moreover, there is (Ab) for the category of abelian groups (morphisms are group homomorphisms), (Vect_K) for the category vector spaces over a field K (morphisms are linear maps).

Note that there exists a forgetful functor from (gps) to (sets) which forgets the underlying group structure.

Suppose C is a category and $A, B \in Ob(C)$. We define the **product** $A \times B$ as an object (also called $A \times B$), together with a pair of maps $A \times B \xrightarrow{\pi_A} A$ and $A \times B \xrightarrow{\pi_B} B$, such that $\forall T \in Ob(C)$ and any maps $\alpha : T \to A, \beta : T \to B$, $\exists! \phi : A \times B \to T$ so that the following diagram commutes:



The **sum** of objects in a category is defined similarly. For any $A, B \in Ob(C)$, define $A + B$ to be a triple:

- $A + B \in Ob(C)$
- and morphisms $i_A : A \to A + B, i_B : B \to A + B$

such that $\forall T \in Ob(C)$, and morphisms $\alpha : A \to T, \beta : B \to T$, $\exists! \phi : A + B \to T$ so that the following diagram commutes: Note that the sum and product in the category of vector spaces are the same thing, the ordinary direct sum.

Let C, D be any two categories. then define a **covariant functor** $F : C \to D$ to be a map $Ob(C) \to Ob(D)$ and a map $\forall A, B \in Ob(C)$

$$Mor_C(A, B) \xrightarrow{\phi} Mor_D(F(A), F(B))$$

The requirements are, $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$, $F(id_A) = id_{F(a)}$. A **contravariant functor** is the same but it maps morphisms in the opposite direction, that is, $\forall A, B \in Ob(C)$

$$Mor(A, B) \xrightarrow{\phi} Mor(F(B), F(A))$$

Homology is a covariant functor from topological spaces to abelian groups, and cohomology is a contravariant functor (wow so cool).

Example of a contravariant functor: let $C = Vect_K$, and definte the functor: $F : C \to C$ by $F(V) = V^*$ and $\forall \phi : V \to W \in Mor(V, W)$, it is true that $F(\phi) := {}^t\phi : W^* \to V^* \in Mor(F(W), F(V))$. Here, ${}^t\phi$ is the transpose of ϕ.

Let C be a category, $A \in Ob(C)$, then we can define a functor

$$C \xrightarrow{F} (sets)$$

such that $\forall B \in Ob(C), F_A(B) = Mor(A, B)$ and for all moprhisms $B \xrightarrow{\phi} C$, one gets the map $Mor(A, B) \to Mor(A, C)$ by composing with ϕ. Yoneda's Lemma states that $A \in Ob(C)$ is determined by the functor F_A.

Back to operators. Let $T : V \to V$ be an operator, then the simplest case occurs when T is **diagonalizable**, i.e., $\exists$ direct sum decomposition

$$V = \oplus V_\lambda$$

such that $T(v_\lambda) = v_\lambda$ and $T|_{v_\lambda} = \lambda \cdot id$

what we get: if K is algebraicaly closed, then $\exists$ flag

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V$$

where $T(V_i) \subset V_i$ (each new proper inclusion gives a new basis element). Equivalently, $\exists$ basis $v_1, \ldots, v_n$ for V such that the matrix representation of T is upper triangular.

In the diagonalizable case, $V = \oplus_{\lambda \in K} V_\lambda$ where $V_\lambda = \ker(T - \lambda)$. This is **OK** in general if we replace $\ker(T - \lambda)$ by $\operatorname{gker} T - \lambda$.

5 Wednesday, 9/28/22

Today: finish description of ooperators on finite dimensional vector spaces, start bilinear forms

Coming up: multilinear algebra, more group theory, representation theory

Let V be a vector space over a field K of finite dimension n. Let $T : V \to V$ any operator. We have noted the following sequences of subspaces

$$0 \subset \ker(()T) \subset \ker(()T^2) \subset \cdots \subset V$$

$$V \supset \operatorname{im}(()T) \supset \operatorname{im}()(T^2) \supset \ldots$$

Observe, if $\ker(()T^m) = \ker(()T^{m+1})$, then they all equal $\ker(T^N) \forall N > M \leftrightarrow \operatorname{im}()(T^{m+1}) = \operatorname{im}()(T^m)$.

The **generalized kernel** of T is

$$\begin{aligned} \operatorname{gker}(T) &:= \{v \in V : T^m v = 0 \text{ some } m > 0\} \\ &= \cup \ker(()T^m) = \ker(()T^n) \end{aligned}$$

One can similarly define the **generalized image**

$$\operatorname{gim}(T) = \cap \operatorname{im}()(T^m) = \operatorname{im}(()T^n)$$

In the special case that $\operatorname{gker} T = V$, we say that T is **nilpotent**.

Key fact: If $T : V \to V$ is an operator, we have a $\oplus$ decomoposition

$$\begin{aligned} V &= \operatorname{gker}(T) \oplus \operatorname{gim}(T) \\ &= \ker()(T^n) \oplus \operatorname{im}()(T^n) \end{aligned}$$

Why? First, because $\dim \ker()(T^n) + \dim \operatorname{im}()(T^n) = n$ <u>and</u> $\operatorname{gker}(T) \cap \operatorname{gim}(T) = 0$.

Let V be a dimension n vector space over an algebraically closed field K. $\forall \lambda \in K$, we have the eigenspace

$$V \supset V_\lambda := \ker(()T - \lambda)$$

and indeed

$$T \text{ is diagonalizable} \Leftrightarrow V = \oplus V_\lambda$$

Now define the **generalized eigenspace**

$$gV_\lambda := \text{gker}\,(T-\lambda) = \ker()(T-\lambda)^n$$

Proposition: If $\lambda_1, \dots, \lambda_m \in K$ are distinct, then $gV_{\lambda_1}, gV_{\lambda_2}, \dots, gV_{\lambda_m}$ is linearly independent, i.e, for $v_i \in gV_{\lambda_i}$,

$$v_1 + \dots v_m = 0 \Rightarrow v_i = 0 \ \forall i,$$

to see this, apply $(T-\lambda_m)^n$ a suitable number of times, v_m will be killed and everything else will be mapped to a nonzero vector.

Theorem: For any $T : V \to T$,

$$V = \oplus_\lambda gV_\lambda$$

$$T : gV_\lambda \to gV_\lambda$$

$$T|_{gV_\lambda} = \lambda + \text{ nilpotent}$$

We will prove this by inducting on $\dim V$. For any eigenvalue λ,

$$V = \text{gker}(T-\lambda) \oplus \text{gim}(T-\lambda)$$

The first term is just gV_λ, and the second term is $\oplus_{\mu \neq \lambda} gV_\mu$

Observe that the proof to the above proposition holds because $\forall u \neq \lambda$, the map $T - \mu : gV_\lambda \to gV_\lambda$ is an isomorphism.

Note that if V is infinite dimensional, then <u>none of this holds</u>, e.g., for $V = K[x]$ and $T = \frac{\mathrm{d}}{\mathrm{d}x}$, $\text{gker}(T) = V$ and $\text{gim}(T) = V$.

Claim: If V is a finite dimensional vector space over a field K, and $T : V \to V$ is nilpotent, then

$$\exists V = \oplus V_\alpha$$

$$T(V_\alpha) \subset V_\alpha$$

$$\text{and } \exists e_1, \dots, e_{n_\alpha} \text{ for } V$$

T is the same map as before (which I didn't have time to write down):

$$e_1 \mapsto 0$$
$$e_2 \mapsto e_1$$
$$e_3 \mapsto e_2$$
$$\vdots$$
$$e_{n_\alpha} \mapsto e_{n_\alpha - 1}$$

This can be proved by induction on $\dim V$, start with $\text{im}(()T) \subset V$A, and apply induction hypothesis to $\text{im}(()T)$... same proof in Axler.

In terms of this basis, the matrix representation of T has 0 across its diagonal, and on its superdiagonal (diagnoal directly above main diagonal) has 0s and $1s$, and 0 everywhere else. In matrix form: $\exists$ basis for V such that the matrix map of T is in **Jordan canonical form**.

Sanity check: if V is of dimension n, then $\text{Hom}(V,V) \supset 1, T, T^2, \dots$ and has dimension n^2 so T will eventually be the root to a polynomial after n^2 terms, but we can do better (in n terms), and we have already shown this.

Say $V = \oplus V_\lambda$, and $\dim V_\lambda = m_\lambda$ (say: eigenvalue λ has multiplicity m_λ), then

$$(T-\lambda)\,|_{v_\lambda}^{m_\lambda} = 0,$$

so T satisfies a polynomial $P(T) = 0$, and P is a polynomial of degree $\leq \sum m_\lambda = \dim V$. This minimal polynomial is known as the characteristic polynomial .

6 Friday, 9/30/22

We will wrap up from last time. Let V be a finite-dimensional (of dimension n) vector space over an algebraicaly closed finite field K. Let $T : V \to V$ be any operator. We have the decomposition

$$V = \oplus V_\lambda$$

where $V_\lambda = \text{gker}(T-\lambda)$, i.e., $T(V_\lambda) = V_\lambda$, and $T|_{V_\lambda} = \lambda I +$ nilpotent . Clearly $\dim V_\lambda = m_\lambda$, and this is the **multiplicity** of the eigenvalue λ. m_λ is the number of λs in a diagonal in UT. matrix representing T, that is, $\sum m_\lambda = u$.

Observe,

$$(T-\lambda)\,|_{V_\lambda}^{m_\lambda} = 0$$

$\implies$ set $P(x) = \prod$... he erased. Minimal polynomials, characteristic polynomials, and jordan blocks and normal form and their matrices.

What if K is not algebraically closed? Then $\exists$ algebraically closed field $L \supset K$. Obvserve: given a vector space V over K, we can associate to it a vector space over L. Concretely: choose a basis $v_1, v_2, \dots, v_n$ for V.

$$V = \{c_1 v_1 + \cdots + c_n v)n : c_i \in K\}$$

$$WL = \{\text{“}\quad\text{”} : c_i \in L\}$$

In fact, we can define a functor

$$\text{Vect}_K \leftrightarrow \text{Vect}_L,$$

but we leave this for a further class when we have learned tensor products (exercise: what do you need to prove to show a given construction is a functor?).

Notions like length of a vector or angles between vectors don't make sense over an arbitrary field K, but they are based on a construction that does: inner product / dot product on $\mathbb{R}^n$. Basic idea: Given $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we can define $(x \cdot y) = \sum x_i y_i$.

Let V be a vector space over an arbitrary field K. A bilinear form on V is a map $V \times V \xrightarrow{b} K$ that is linear in each variable separately.

$$\begin{aligned} b(\lambda x, y) &= \lambda b(x,y) \\ b(x+x', y) &= b(x,y) + b(x', y) \\ b(x, \lambda y) &= \lambda b(x,y) \\ b(x, y+y') &= \lambda b(x,y) + b(x,y') \end{aligned}$$

We say that b is **symmetric** if $b(x,y) = b(y,x)\ \forall x, y$. Similarly, we say b is **skew-symmetric** if $b(x,y) = -b(y,x)\ \forall x, y$.

Observe that if $\dim K \neq 2$, then every bilinear form is expressible as a sum of a symmetric and a skew form.

Given $b : V \times V \to K$, then

$$b_1(v_1, w) := \frac{b(v_1, w) + b(w_1, v)}{2}$$

$$b_2(v_1, w)L = \frac{b(v_1, w) - b(w, v_1)}{2}$$

are a unique construction for the above decomposition. Exercise: prove this.

Observe that given a vector space V over K, the set of all bilinear forms

$$B := \{ \text{ bilinear forms } b : V \times V \to K\}$$

is a vector space. We can also define B_{symm} and B_{skew}, and if $\text{char}(K) \neq 2$, then we have

$$B = B_{\text{symm}} \oplus B_{\text{skew}}$$

Basic fact: $\dim B = (\dim B)^2$.

Suppose $b : V \times V \to K$, the fact that b is linear in the second variable $\implies$ we get a map

$$\begin{aligned} V &\to V^* \\ v &\mapsto b(v, \cdot) \end{aligned}$$

Because b is linear in the first variable, the map $\tilde{b}$ is linear, that is, the map

$$\text{rank}(b) := \text{rank}(\tilde{b})$$

We say that b is **non-degenerate** if $\text{rank}(b) = n = \dim V$, i.e., $\forall v \neq 0 \in V\ \exists w \in V : b(v, w) = 0$

We get a map

$$\begin{aligned} B(V) &\xrightarrow{\cong} \text{Hom}(V, V^*) \\ b &\mapsto \tilde{b} \end{aligned}$$

that is an isomorphism. From this isomoprhism, we get $\dim B = n^2$. (note $b(v,w) = \left(\tilde{b}\,(v)\right)(w)$)

We can prove this directly: choose a basis $e_1, \dots, e_n$ for V, and for any $v, w \in V$, let

$$v = c_1 e_1 + \cdots + c_n e_n$$
$$w = d_1 e_1 + \cdots + d_n e_n$$
$$b(v,w) = \sum_{i,j} c_i d_j b(e_i, e_j)$$

we get an isomorphism $B \cong K^{n^2}$. Note that the $b(e_i, e_j)$ can be identified as a matrix representation of b w.r.t. the basis $e_1, \dots, e_n$, that is, we have

$$b(v,w) = (c_1, \dots, c_n) M \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

where M is an $n \times n$ matrix.

We can define a **trilinear form** (or for any number of variables) to be a map $V \times V \times V \to K$ that is linear in each factor. Similar as before, we can let $T = \{\text{trilinear forms on } V\}$, and similarly define T_{symm}, and to define T_{skew} as bilinear forms whose sign change by -1 raised to the parity of the permutation of the elements.

Question: is it true that $T = T_{\text{symm}} \oplus T_{\text{skew}}$?

Hint: no

Suppose we have a vector space V of dimension n over K, and a bilinear form $b : V \times V \to K$. If $U \subset V$ is a subspace, then define the **orthogonal complement** of U as

$$\begin{aligned} U^{\perp} &= \{w \in V : b(w,u) = 0 \ \forall u \in U\} \\ &= \operatorname{Ann}\left(\tilde{b}\,(u)\right) \end{aligned}$$

where $\tilde{b} : V \to V^*$ is defined the same as before.

Note that in the "normal" setting of $\mathbb{R}^n$ where the inner product/bilinear form is the dot product, it is true that for any subspace U,

$$V = U \oplus U^{\perp}$$

Note that this is not true in general. For example, if $V = \mathbb{C}^2$, define $b(x,y) = x_1 y_1 + x_2 y_2 \in \mathbb{C}$. Then for the subspace $U = \langle (1,i) \rangle$, it is true that $U^{\perp} = U$, so while their dimension add up to the dimension of V, they are clearly not disjoint so the previous direct sum decomposition of V does not hold. We highlight this dimension correspondence:

$$\operatorname{Ann}(u) = \{v \in V : \ U(v) = 0 \ \forall u \in U\}$$

The claim is that $\dim \operatorname{Ann}(u) + \dim U = n$ We proved this before (you just need to extend a basis of U).
To see another more general counterexample, let $V = K^2$ for any field K, and $b(x, y) = x_1 y_2 - x_2 y_1$, then for any one dimensional subspace $U \subset V$ satisfies $U = U^{\perp}$.